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CO-ABSOLUTES OF $\beta N\backslash N$

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It is shown that "every Parovičenko space is co-absolute with $\beta N\backslash N$" does not imply CH and in turn is not implied by $c<2^{\omega_1}$. We also obtain a characterization of the above statement in terms of the non-existence of certain trees.

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1. Introduction

Let us call a space X a *Parovičenko* space if

(i) X is a compact zero-dimensional space of π-weight c with no isolated points,

(ii) every two disjoint open F_σ's have disjoint closures, and

(iii) every non-empty G_δ has non-empty interior.

(Recall that a π-base for a space X is a collection of non-empty open subsets, $\mathcal{B}$, such that for each non-empty open $U\subset X$, $U\supset B$ for some $B\in\mathcal{B}$. The π-weight of a space is the minimum cardinality of a π-base).

Perhaps the best known example of a Parovičenko space is $\beta N\backslash N$ (where βX is the Čech–Stone compactification of X and N is the discrete space of natural numbers). Also, X^* $(=\beta X\backslash X)$ is a Parovičenko space whenever X is a zero-dimensional, locally compact, σ-compact space with weight c [7]. It has been shown that the continuum hypothesis (abbreviated CH and GCH is the generalized continuum hypothesis) is equivalent to the statement that all Parovičenko spaces of weight c are homeomorphic ([2], [5]).

The *absolute* (see [17]) of a regular space X is the unique (up to homeomorphism) extremally disconnected space, EX, which can be mapped by a perfect irreducible map onto X. Spaces X and Y are called *co-absolute* if EX is homeomorphic to EY. In [16], R.G. Woods showed the following result.

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1.1. Theorem. CH *implies that* X^* *is co-absolute with* N^* *for each locally compact,* σ*-compact* X *with weight* c.

This has been improved by Broverman and Weiss, [2] who showed

1.2. Theorem. CH *implies that every Parovičenko space is co-absolute with* N^*.

However the CH assumption cannot be removed completely in either result. Indeed, from the assumption of MA+$\neg$CH, (MA is Martin's Axiom, see [10]), it can be shown that $(\omega \times 2^{\omega_1})^*$ is not co-absolute with N^*. This follows by a result in [5] where it is shown that $(\omega \times 2^{\omega_1})^*$ is covered by nowhere dense G_{ω_1}-sets. It is fairly well-known that MA+$\neg$CH implies that there are *no* non-empty nowhere dense G_{ω_1}-subsets of N^*.

Partial converses to 1.2 have been obtained, first by Broverman and Weiss and very recently improved by Van Mill and Williams.

1.3. Theorem [11]. *If every Parovičenko space* (*even of weight* c) *is co-absolute with* N^* *then* $c < 2^{\omega_1}$.

(Broverman and Weiss [2] had that $c < 2^{<c}$).

The main purpose of this paper is to show that the converse to each of 1.1, 1.2 and 1.3 is false. In so doing we shall also prove

1.4. Theorem. *If the cofinality of* c (*abbreviated* $cf(c)$) *is* ω_1 *then all Parovičenko spaces are co-absolute.*

We do not know if the converse to 1.4 is true but in Section 4 we obtain an equivalence of 'all Parovičenko spaces are co-absolute' in terms of the non-existence of certain trees.

Let X be a topological space with no isolated points. The *Novák number* of X, $n(X)$, is the minimum cardinality of a family of nowhere dense sets covering X. The Novák number of N^* is studied in detail by Balcar, Pelant and Simon in [1]. This paper has benefitted greatly by their results. In [11], the weak Novák number of X, $wn(X)$, is introduced and is defined to be the minimum cardinality of a family of nowhere dense sets whose union is dense. For a Parovičenko space X, it turns out that $wn(X)$ is equal to the cardinal invariant $\kappa(X)$ introduced in [1].

The two cardinal invariants, $wn(X)$ and $n(X)$, seem to be the only 'topological' tools available for distinguishing the absolutes of Parovičenko spaces. In Section 3 we show that all Parovičenko spaces with weak Novák number ω_1 are co-absolute. In the final section we give an example to show that this result will not generalize to all Parovičenko space with weak Novák number ω_2.

2. Preliminaries

In this section we introduce some notation and tools which we shall require. Most of our notation and terminology is standard, in particular an ordinal is the set of its predecessors and a cardinal is an initial ordinal. We borrow an idea of Scott Williams which is to study Parovičenko space by investigating π-bases which, when ordered by reverse inclusion, form trees. The idea is also very similar to the concept of a base matrix introduced in [1] and studied extensively for N^*. We shall find it very useful to restate the results of [1] in terms of trees and prove the easy generalizations from N^* to arbitrary Parovičenko spaces.

2.1. Definition. A partially ordered set $(T, <)$ is a *tree* if for each $t \in T$, $\{s \in T: s < t\}$ is well-ordered. For $t \in T$, the order type of $\{s \in T: s < t\}$ is denoted $o(t)$ and is called the *order* of t. The αth *level* of T, T_α, is defined as $\{t \in T: o(t) = \alpha\}$. The *height* of T, $ht(T)$, is defined to be $\min\{\alpha: T_\alpha = \emptyset\}$. For convenience we shall require that if $t \neq t' \in T$ with $o(t)$ a limit ordinal then $\{s \in T: s < t\} \neq \{s \in T: s < t'\}$. T is said to be α-*branching* if for each $t \in T$, $|\{s \in T: s > t \wedge o(s) = o(t)+1\}| = \alpha$. For a cardinal λ, T is λ-*complete* if each chain of cardinality λ has a supremum in T. For an ordinal κ and a set S, $({}^{<\kappa}S, <)$ is the tree obtained by ordering ${}^{<\kappa}S = \bigcup_{\alpha<\kappa} {}^{\alpha}S$ by extension. As is usual, for an element $t \in {}^{<\kappa}S$ and $s \in S$, $t^\frown(s)$ denotes the extension of t which takes the value s at $o(t)$. Similarly for a tree T and $t \in T$, if $\alpha \in o(t)$ then $t|_\alpha$ denotes the element $t' < t$ such that $o(t') = \alpha$. A *subtree* of T is a subset $T' \subset T$ such that for $t \in T'$, $\{s \in T: s < t\} \subset T'$.

The next result is due to Scott Williams and shows clearly the reason for our interest in trees. It is proved in [15] in the language of posets,

2.2. Theorem. *Every Parovičenko space has a π-base of regular open sets which, when ordered by reverse inclusion, form an ω-complete tree with height and cardinality at most c. We may also insist that at non-limit levels the sets are clopen.*

Proof. Let X be a Parovičenko space with π-base $\mathscr{B} = \{B_\beta: \beta \in c\}$ of clopen sets. We shall choose a π-base for X indexed by a sub-tree of ${}^{<c}c$. Let $[\emptyset] = X$ and $T_0 = \{\emptyset\}$ where $\emptyset \in {}^{<c}c$ is the empty function. Suppose that $\alpha < c$ and for $\gamma < \alpha$ we have recursively chosen sub-trees T_γ of ${}^{<\gamma}c$ and, for $t \in T_\gamma$, corresponding regular open subsets, $[t]$, of X exactly as we shall define T_α and $[t]$ for $t \in T_\alpha \backslash (\bigcup_{\gamma<\alpha} T_\gamma)$.

Case 1: α a limit. Let $T_\alpha = \bigcup_{\gamma<\alpha} T_\gamma \cup \{t \in {}^{\alpha}c: t|\gamma \in T_\gamma$ for each $\gamma < \alpha$ and $\operatorname{int} \bigcap_{\gamma<\alpha} \operatorname{cl}_x[t|_\gamma] \neq \emptyset\}$. For $t \in T_\alpha \backslash (\bigcup_{\gamma<\alpha} T_\gamma)$ let $[t] = \operatorname{int} \bigcap_{\gamma<\alpha} \operatorname{cl}_X[t|_\gamma]$.

Case 2: $\alpha = \gamma + 1$. Take $\kappa \leqslant c$ and let $\{t_\xi: \xi < \kappa\}$ be a faithful indexing of $T_\gamma \backslash (\bigcup_{\delta<\gamma} T_\delta)$. Recursively, for each $\beta < c$ such that $|\{\xi < \kappa: B_\beta \cap [t_\xi] \neq \emptyset\}| = c$, choose a $\xi_\beta < \kappa$ not already chosen with $B_\beta \cap [t_{\xi_\beta}] \neq \emptyset$. Observe that each non-empty open subset of X contains cellular families of clopen sets of size c since non-empty G_δ's have non-empty interior. So for each $\xi < \kappa$, we may choose a maximal cellular family of size c of clopen subsets of $[t_\xi]$ such that both the following conditions are fulfilled. First, if $\xi = \xi_\beta$ for some $\beta < c$ then some member of this family is contained in $B_\beta \cap [t_{\xi_\beta}]$ and secondly, if $B_\gamma \cap [t_\xi] \neq \emptyset$ then some member is contained in this intersection. Label this family by $\{[t_\xi \frown (\delta)]: \delta \in c\}$.

Now for some limit ordinal $\kappa \leq c$, our induction stops for the reason that for each $t \in {}^\kappa c$ with $t|_\gamma \in T_\gamma$ for $\gamma < \kappa$, $\operatorname{int} \bigcap_{\gamma<\kappa} \operatorname{cl}[t|\gamma] = \emptyset$. For suppose otherwise, that is, for some $t \in {}^c c$, $\operatorname{int} \bigcap_{\alpha<c} \operatorname{cl}[t|\alpha] \neq \emptyset$. However, if $\gamma < c$ with $B_\gamma \subset \bigcap_{\alpha<c} [t|\alpha]$, then at stage $\alpha = \gamma + 1$, we ensured that, for some $\delta \in c$, $[t|\gamma \frown (\delta)] \subset B_\gamma \cap [t|\gamma]$. Therefore it cannot be the case that $B_\gamma \subseteq [t|_{\gamma+2}]$. Let $T = \bigcup_{\alpha<\kappa} T_\alpha$. It is clear that T is ω-complete because X is a Parovičenko space and we used clopen sets at non-limits.

To show that $\{[t]: t \in T\}$ is a π-base for X it suffices to show that, for each $\gamma < c$, there is a $t \in T$ with $[t] \subset B_\gamma$. (Note that it may be the case that $\kappa < c$, hence we cannot simply claim that B_γ is refined at stage $\gamma + 1$). Let $\gamma < c$ and for $t \in T$ define

$$g(t) = \min\{\alpha: \text{for all } s \in T_\alpha \cap \{s: s > t\}, [s] \cap B_\gamma = \phi\}.$$

Let $\alpha_0 = \min\{g(t): [t] \cap B_\gamma \neq \phi\}$ and choose t_0 so that $[t_0] \cap B_\gamma \neq \phi$ and $g(t_0) = \alpha_0$.

Fact: for each $t \geq t_0$ and $\delta < \alpha_0$ with $[t] \cap B_\gamma \neq \phi$ and $o(t) \leq \delta$ there is an $s \in T_\delta$ with $t < s$ and $[s] \cap B_\gamma \neq \phi$. Indeed, by the minimality of α_0, for each such t, $g(t) \geq g(t_0) = \alpha_0$ and the fact follows.

Now, since $\operatorname{int} \cap \{\operatorname{cl}[t]: t \in b\} = \phi$ for each maximal chain b of T, each clopen set meets at least two members of T_α for some level $\alpha < \kappa$. We may therefore recursively choose, for $n \in \omega$, antichains $\{t_f: f \in {}^n 2\}$ of T so that, for each $f \in {}^n 2$, $B_\gamma \cap [t_f] \neq \phi$, $[t_f]$ is clopen, $t_0 < t_{f|_k} < t_f$ for $k < n$ and finally $o(t_f) = o(t_g)$ for all $g \in {}^n 2$. This last condition follows from the above Fact. Let $\alpha = \sup\{o(t_f): f \in {}^{<\omega} 2\}$ and let, for $f \in {}^\omega 2$, $t_f = \sup\{t_{f|_n}: n \in \omega\} \in T_\alpha$. Since non-empty G_δ's of X have non-empty interior, $[t_f] \cap B_\gamma \neq \phi$ for $f \in {}^\omega 2$.

The above result is proven for N^* in the context of base matrices in [1]. For the remainder of the paper let us agree that a *πn-tree* for a space X is a π-base for X which forms a tree when ordered by reverse inclusion. For a space X possessing a πn-tree let $\kappa(X)$ be the minimum height of a πn-tree for X. It can be easily shown that for Parovičenko spaces our definition of $\kappa(X)$ coincides with the definition given in [1]. The next result is very similar to 1.7 of [15] but is expressed in different terminology. It is crucial to our main results so we include a proof for completeness.

2.3. Theorem. *For a Parovičenko space X, $\omega_1 \leq \kappa(X) = wn(X)$ and X has a πn-tree which is c-branching, ω-complete, has height $\kappa(X)$ and clopen sets at non-limit levels.*

Proof. Since non-empty G_δ's of X have non-empty interior, it is immediate that ω is less than both $\kappa(X)$ and $wn(X)$. Let $\kappa = \kappa(X)$ and choose a πn-tree, T, of height κ.

For each $\alpha < \kappa$, let $D_\alpha = \operatorname{cl}_X\{\bigcup\{t: t \in T_\alpha\}\} \backslash \bigcup\{t: t \in T_\alpha\}$.

Clearly each D_α is nowhere dense. Let $t \in T$ and $\alpha > o(t)$ be such that $\{s \in T: t < s$ and $o(s) = \alpha\}$ is infinite. (The fact that there is such an α follows from the facts that there are no isolated points and T is a π-base). Since X is compact, $\mathrm{cl}_X\, t \cap D_\alpha \neq \emptyset$. It follows that $\bigcup_{\alpha<\kappa} D_\alpha$ is dense as T is a π-base. Hence $wn(X) \leqslant \kappa(X)$.

Conversely, suppose $\bigcup_{\alpha<\kappa} D_\alpha$ is dense in X with each D_α nowhere dense. In the proof of 2.2 at stage $\alpha = \gamma + 1$ we now insist that each member of the chosen cellular family is disjoint from D_γ. We claim then that our induction stops by stage κ. Indeed, let $t \in {}^\kappa c$ with $t|\alpha \in T$ for each $\alpha < \kappa$. Then clearly $\bigcap_{\gamma<\kappa} \mathrm{cl}_X[t|\gamma] \cap \bigcup_{\alpha<\kappa} D_\alpha = \emptyset$, hence $\mathrm{int} \bigcap_{\gamma<\kappa} \mathrm{cl}_x[t|\gamma] = \emptyset$ as $\bigcup_{\alpha<\kappa} D_\alpha$ is dense. Therefore $\kappa(X) \leqslant wn(X)$. The construction of this πn-tree satisfies the above requirements. □

There is a non-trivial relationship between $wn(X)$ and $n(X)$ whenever $wn(X) < c$ for many spaces X possessing a πn-tree. It is clear that $wn(X) \leqslant n(X)$ but the following result of Kulpa and Szymański, restated here in terms of πn-trees, is non-trivial and useful.

2.4. Theorem [9]. *Let X be a space having a πn-tree, T, of height κ and suppose, for each $t \in T$, there is a cellular family, $\{t(\xi): \xi < \kappa^+\}$ of non-empty open subsets of t, then $n(X) \leqslant \kappa^+$.*

Proof. For each $\eta < \kappa^+$, define $D_\eta = X \backslash \bigcup \{t(\xi): \eta < \xi < \kappa^+,\ t \in T\}$. Clearly D_η is closed. To see that it is nowhere dense, observe that, for each $t \in T$ and $\xi > \eta$, $t(\xi) \subset t$ and $t(\xi) \cap D_\eta = \emptyset$. Finally, let $x \in X$. For each $\alpha < \kappa$ there is at most one $t_\alpha \in T_\alpha$ and one $\xi_\alpha < \kappa^+$ with $x \in t_\alpha(\xi_\alpha)$. Let $\eta = \sup\{\xi_\alpha: \alpha < \kappa\} < \kappa^+$. Since $x \notin t(\xi)$ for each $t \in T$ and $\eta < \xi < \kappa^+$, $x \in D_\eta$. Therefore $X = \bigcup_{\eta<\kappa^-} D_\eta$. □

For a tree T, we shall call a maximal chain $b < T$, a *long chain* if $o(b) = ht(T)$. The next result appears in [1] for the case $X = N^*$.

2.5. Lemma. *For a Parovičenko space X, $wn(X) = n(X)$ iff X has a πn-tree T of height $wn(X)$ with no long chains.*

Proof. First suppose that $wn(X) = n(X) = \kappa$ and we have $X = \bigcup_{\alpha<\kappa} D_\alpha$ with each D_α nowhere dense. As in 2.3, we can construct a πn-tree T such that for $\alpha < \kappa$ and $t \in T_{\alpha+1}$, $t \cap D_\alpha = \emptyset$. Now suppose $b \subset T$ is a maximal chain. Since $\bigcap\{\mathrm{cl}_x\, t: t \in b\} \neq \emptyset$, it must meet D_α for some $\alpha < \kappa$. It follows that $o(b) < \kappa$ since $b \cap T_{\alpha+1}$ must be empty.

Conversely, let T be a πn-tree without long chains and $ht(T) = \kappa = wn(X)$. For each $\alpha < \kappa$, let $D_\alpha = \mathrm{cl}(\bigcup\{t: t \in T_\alpha\}) \backslash \bigcup\{t: t \in T_\alpha\}$. Observe that $X_\alpha = \mathrm{cl}(X \backslash \mathrm{cl}(\bigcup\{t: t \in T_\alpha\})$ is a space possessing a πn-tree (of course X_α may be empty). From 2.4 we may choose nowhere dense sets $\{E_{\alpha\gamma}: \gamma < \kappa\}$ such that $X_\alpha = \bigcup\{E_{\alpha\gamma}: \gamma < \kappa\}$. We claim that $X = \bigcup_{\alpha<\kappa} \{D_\alpha \cup \bigcup_{\gamma<\kappa} E_{\alpha\gamma}\}$. For if this is not the case then there is an $x \in X$ such that for each $\alpha < \kappa$, $x \in t_\alpha$ for some $t_\alpha \in T_\alpha$. However $\{t_\alpha: \alpha < \kappa\}$ is then a long chain of T. □

It is well-known that for a compact space X, EX is homeomorphic to the Stone space of the boolean algebra of regular open subsets of X, $RO(X)$ (see [17]). Therefore X and Y are co-absolute iff the boolean algebras $RO(X)$ and $RO(Y)$ are isomorphic. The following consequence of this and 2.2 was first observed by Williams [14].

2.6. Theorem. *Let X and Y be Parovičenko spaces. Then X and Y are co-absolute iff they possess isomorphic πn-trees.*

Finally let us record the following result.

2.7. Lemma. *For a compact space X, $wn(X) = wn(EX)$ and $n(X) = n(EX)$.*

Proof. This is a straightforward consequence of the fact that there is a continuous irreducible map from EX to X. $\square$

3. Main results

The main result of this section is the proof of Theorem 1.4. We also present various models to show, for instance, that the converse of 1.3 is false and that all Parovičenko spaces of the form $\beta X - X$ for σ-compact, locally compact X may be co-absolute without all Parovičenko spaces being co-absolute.

3.1. Theorem. *All Parovičenko spaces with weak Novák number equal ω_1 are co-absolute.*

Proof. Let X be a Parovičenko space with $wn(X) = \omega_1$. By 2.3, X has a πn-tree T with $ht(T) = \omega_1$ and which is c-branching and ω-complete. Therefore T is isomorphic to ${}^{<\omega_1}c$. By 2.6 the proof is complete. $\square$

The following corollary was first proven by Williams [15].

3.2. Corollary. *For each $\omega_1 \leq \kappa \leq c$, $(\omega \times 2^\kappa)^*$ is co-absolute with $(\omega \times 2^{\omega_1})^*$.*

Proof. As remarked in the introduction, $(\omega \times 2^\kappa)^*$ is a Parovičenko space for each $\omega_1 \leq \kappa \leq c$. For $\alpha \in \omega_1$, let $D_\alpha = \{f \in 2^\kappa : f(\beta) = 0 \text{ for } \alpha < \beta < \omega_1\}$. Clearly each D_α is nowhere dense and one can check that $\bigcup_{\alpha<\omega_1} D_\alpha$ is dense in 2^κ. For each $\alpha \in \omega_1$, let D_α^* be the nowhere dense subset of $(\omega \times 2^\kappa)^*$ defined as $\mathrm{cl}_{\beta(\omega\times 2^\kappa)}(\omega \times D_\alpha) \cap (\omega \times 2^\kappa)^*$. If, for $n \in \omega$, C_n is a basic clopen subset of 2^κ, then for some $\alpha \in \omega_1$, $D_\alpha \cap C_n \neq \emptyset$ for each $n \in \omega$. Therefore $D_\alpha^* \cap \mathrm{cl}_{\beta(\omega\times 2^\kappa)}[\bigcup_{n\in\omega} \{n\} \times C_n] \neq \emptyset$ and $wn((\omega \times 2^\kappa)^*) = \omega_1$. $\square$

We remark that $wn((\omega\times 2^\kappa)^*)=\omega_1$ for $\kappa\geqslant\omega_1$ also follows easily from results in [5]. The following result was proven for the case $X=N^*$ in [1].

3.3. Theorem. *$\omega_1\leqslant wn(X)\leqslant cf(c)$ for each Parovičenko space X.*

Proof. Let X be a Parovičenko space. It follows easily that the density of X is c by the facts that $\pi w(X)=c$ and that there are cellular families of size c of non-empty open sets. Let $D=\{d_\alpha\colon \alpha<c\}$ be a dense subset of X. Let $\{\lambda_\alpha\colon \alpha<cf(c)\}$ be a cofinal sequence in c. Take, for each $\alpha<cf(c)$, the nowhere dense set $D_\alpha=\mathrm{cl}_X\{d_\gamma\colon \gamma<\lambda_\alpha\}$. Since $D\subset\bigcup_{\alpha<cf(c)}D_\alpha$, $wn(X)\leqslant cf(c)$. □

3.4. Proof of the Theorem 1.4. By 3.3, if $cf(c)=\omega_1$ then all Parovičenko spaces have weak Novák number ω_1. Hence all Parovičenko spaces are co-absolute by 3.1. □

We have as an immediate corollary that the converses to 1.1 and 1.2 are false.

3.5. Corollary. *It is consistent with* $\neg$CH *that all Parovičenko spaces are co-absolute.*

Proof. Let M be a model of ZFC$+2^\omega=\aleph_{\omega_1}$.

The next two results also appear in [15]:

3.6. Theorem. *$wn(X^*)\leqslant wn(N^*)$ for each locally compact σ-compact zero-dimensional space X.*

Proof. Let $X=\bigcup\{X_n\colon n\in N\}$, where the X_n's are pairwise disjoint clopen compact subsets of X. Define $f\colon X\to N$ by $f(x)=n$ iff $x\in X_n$. Let $\beta f=\beta X\to\beta N$ be the Čech–Stone extension and let $g=\beta f|X^*$. Observe that if $U\subset X^*$ is clopen, then $g(U)\subset N^*$ is clopen. This is because if $U=C\cap X^*$ for a clopen $C\subset\beta X$, then $g(U)=\mathrm{cl}_{\beta N}\{n\colon C\cap X_n\neq\emptyset\}\backslash N$. Therefore, if $D\subset N^*$ is nowhere dense then $g^\leftarrow(D)\subset X^*$ is nowhere dense and if $\bigcup\{D_\alpha\colon \alpha<\kappa\}\subset N^*$ is dense then $\bigcup\{g^\leftarrow(D_\alpha)\colon \alpha<\kappa\}\subset X^*$ is dense. □

3.7. Corollary. *If $wn(N^*)=\omega_1$, then all growths of σ-compact locally compact zero-dimensional spaces of weight c are co-absolute.*

We just remark that the zero-dimensional assumption can be dropped. The following result is proven in [1] but we include our proof using the concept of $wn(N^*)$. Let $\lambda=\inf\{|B|\colon B\subset {}^NN$ is unbounded$\}$ ($B\subset {}^NN$ is said to be *unbounded* if for each $g\in {}^NN$ there is a $b\in B$ with $|\{n\colon b(n)>g(n)\}|=\omega$).

3.8. Theorem. $wn(N^*)\leqslant\lambda$.

Proof. Let $B \subset {}^N N$, with $|B| \leq \lambda$, be unbounded. For each $b \in B$, choose a maximal almost disjoint family $\mathcal{A}_b \subset P(N)$ such that for $A \in \mathcal{A}_b$, $|\{n: b(n) > f_A(n)\}| < \omega$ (where $f_A: N \to A$ is the order-preserving enumeration of $A \in P(N)$). Let $D_b = N^* \backslash \bigcup \{\mathrm{cl}_{\beta N} A: A \in \mathcal{A}_b\}$ for $b \in B$. Each D_b is nowhere dense and it will suffice to show that $\bigcup(D_b: b \in B)$ is dense in N^*. Take $A \subset N$ and define $g \in {}^N N$ by $g(n) = f_A(n^2)$, $n \in N$. By the choice of B, there is a $b \in B$ with $|\{n: b(n) > g(n)\}| = \omega$. Suppose that $\{A_i: i < k < \omega\} \subset \mathcal{A}_b$ and that $|A \backslash \bigcup_{i<k} A_i| = m < \omega$. We may choose $n > k + m$ with $b(n) > g(n)$ and $b(n) < f_{A_i}(n)$ for $i < k$. Now observe that, for each $i < k$, $|A_i \cap \{j: j < b(n)\}| < n$ by definition of f_{A_i} whereas $|A \cap \{j: j < b(n)\}| \geq n^2$, since $b(n) \geq f_A(n^2)$. The fact that $n^2 > n \cdot (k+m) > |\bigcup_{i<k} A_i \cap \{j: j < b(n)\}| + m$ contradicts that $|A \backslash \bigcup_{i<k} A_i| = m$. Therefore A is not almost contained in a finite union from $\mathcal{A}_b$ so $\mathrm{cl}_{\beta N} A \cap D_b \neq \emptyset$. $\square$

The remaining results of this section require some basic forcing constructions and, with perhaps the exception of 3.11 which is proven in [1], all of our constructions are well-known and straightforward. For this reason we will not include any proofs of the familiar properties of the models so constructed. We refer the reader to Kunen's book [10] for background and proofs.

Let M be a countable model of GCH. Take P_0, P_1 and P_2 to be the following posets defined in M.

a) P_0 is the poset constructed in VIII.6 of $[K]$ to yield the consistency of $\mathrm{MA} + 2^\omega = 2^{\omega_1} = \omega_2$.

For a cardinal κ, $Fn(I, J, \kappa) = \{p: |p| < \kappa \wedge p$ is a function from I into $J\}$ with $p \leq q$ iff $q \subseteq p$.

b) Let $P_1 = F_n(\omega_3, 2, \omega_1)$ and $P_2 = F_n(\omega_2, 2, \omega)$.

Let G_0, G_1 and G_2 be generic filters of P_0, P_1 and P_2 respectively such that $G_0 \times G_1 \times G_2$ is a generic filter of the product $P_0 \times P_1 \times P_2$ (see VIII.1.4 of [10]). We remind the reader that if G is a generic filter of a poset P of M then $M[G]$ is the unique smallest model of ZFC containing M and $\{G\}$. If φ is a statement then $M[G] \models \varphi$ means that φ is valid in $M[G]$.

3.9. Lemma. $M[G_0] \models \mathrm{MA} + 2^\omega = 2^{\omega_1} = \omega_2 + 2^\kappa = \kappa^+$ *for* $\kappa > \omega_1$ (VIII.g of [10]).

3.10. Lemma. $M[G_2] \models 2^\omega = 2^{\omega_1} = \omega_2$ *and* $\lambda = \omega_1$ *where* λ *is as in* 3.8 (*this is essentially* VIII.2.3 *and* VIII.2.6 *of* [10]).

3.11. Lemma. *Let* $i \in \{0, 2\}$ *and let* X *be a Parovičenko space of* $M[G_i]$ *having a* πn*-tree isomorphic to* ${}^{<\omega_2}\omega_2 \cap M[G_i]$. *Then, in* $M[G_i][G_1]$, $n(X) = \omega_3$, $2^\omega < 2^{\omega_1}$ *and* $P(N)$ *is unchanged, from* $M[G_i]$ *to* $M[G_i][G_1]$.

3.12. Theorem. *It is consistent that* $(\omega \times 2^{\omega_1})^*$ *is not co-absolute with* N^* *and* $2^\omega < 2^{\omega_1}$.

Proof. A well-known consequence of MA+$\neg$CH, Booth's theorem, is that non-empty G_{ω_1}-sets in N^* have non-empty interior. It follows easily that $M[G_0]\models N^*$ has a πn-tree isomorphic to ${}^{<\omega_2}\omega_2 \cap M[G_0]$. Hence, by 3.11, $M[G_0\times G_1]\models n(N^*)=\omega_3$. Therefore, by 2.4, $wn(N^*)>\omega_1$ whereas $wn((\omega\times 2^{\omega_1})^*)=\omega_1$. $\square$

3.13. Theorem. *It is consistent that* $2^\omega<2^{\omega_1}$, $wn(N^*)=\omega_1$ *and not all Parovičenko spaces are co-absolute.*

Proof. It is proved in [11] (or see 4.1) that if $2^\omega=2^{\omega_1}$ then there is a Parovičenko space with non-empty G_{ω_1}-sets having non-empty interior. It follows then from 2.2, that if $2^\omega=2^{\omega_1}=\omega_2$ then there is a Parovičenko space, X, with a πn-tree isomorphic to ${}^{<\omega_2}\omega_2$. In particular there is such an X in $M[G_2]$ by 3.10. Therefore in $M[G_2][G_1]$ there is a Parovičenko space X with $n(X)=\omega_3$ and, as above, $wn(X)>\omega_1$. However $\lambda=\omega_1$ in $M[G_2][G_1]$, so $wn(N^*)=\omega_1$ by 3.8. $\square$

It has now been shown that the converse to 1.3 is false and that 3.7 cannot be strengthened to include all Parovičenko spaces even with the assumption of $2^\omega<2^{\omega_1}$.

4. Parovičenko trees

The original purpose of this section was to show that the converse to 1.4 was false. We have been unable to do that but an equivalence of "all Parovičenko spaces are co-absolute" has been found in terms of the existence of certain trees. The main idea is the converse to 2.2. The next two results were inspired by the construction in [11].

4.1. Theorem. *If T is an ω-complete branching tree of cardinality c with $ht(T)\leq c$, then there is a Parovičenko space $X(T)$ having a πn-tree isomorphic to T.*

The only difficulty, of course, is to satisfy condition (ii) of the definition of a Parovičenko space. Before proving 4.1 we shall first prove a lemma. If f is a continuous map from X onto Y and T is a tree of regular open subsets of Y, then $f':T\to RO(X)$ will denote the map defined as follows: for $t\in T$, $o(t)$ not a limit, $f'(t)=f^{\leftarrow}(t)$ and for $o(t)$ a limit, $f'(t)=\operatorname{int}\bigcap_{\alpha<o(t)}\operatorname{cl} f'(t\restriction\alpha)$.

4.2. Lemma. *Let X be a compact zero-dimensional space of π-weight c having a πn-tree T with clopen sets at non-limit levels. There is a Parovičenko space $P(X)$, a map $\varphi_x: P(X)\to X$ and a ω-complete πn-tree T' for $P(X)$ such that $\varphi'_x(T)$ is a subtree of T' and for $t'\in T'\setminus T$ there is a $t\in T_\sigma$ (= ω-completion of T in T') with $|\{s\in T': t\leq s\leq t'\}|=\omega$.*

Proof. Let $X_0=X$. We define an inverse limit system $(x_\alpha, f_{\alpha\beta}, \omega_1)$ as follows. If

$\alpha=\gamma+1$, let $X_\alpha=(\omega\times X_\gamma)^*$. The projection map $\Pi:\omega\times X_\gamma\to X_\gamma$ extends to the map $\Pi^*:X_\alpha\to X_\gamma$. For $\beta\leq\gamma$, define $f_{\alpha\beta}=f_{\gamma\beta}\circ\Pi^*$ (where $f_{\gamma\gamma}=\mathrm{id}_{X_\gamma}$). If α is a limit, let

$$Y_\alpha=\underline{\lim}\{X_\beta:\beta<\alpha\}$$

and define $X_\alpha=(\omega\times Y_\alpha)^*$, $\Pi^*_\alpha:X_\alpha\to Y_\alpha$ and $\{f_{\alpha\beta}:\beta\leq\alpha\}$ as above.

Let

$$P(X)=X_{\omega_1}=\underline{\lim}\{X_\alpha:\alpha\in\omega_1\}$$

and

$$\varphi_x=f_{\omega_1 0}=\underline{\lim}\{f_{\alpha 0}:0<\alpha<\omega_1\}.$$

Recall that $P(X)$ has a base of clopen sets of the form $f^{\leftarrow}_{\omega_1\alpha}[C]$ where $\alpha\in\omega_1$ and C is a clopen subset of X_α. Since $cf(\omega_1)=\omega_1>\omega$ and each X_α for $\alpha>0$ is a Parovičenko space, $P(X)$ is a Parovičenko space. For convenience, if given, $\{U_n:n\in\omega\}$, subsets of X_γ, for $\gamma\in\omega_1$, when we write $(\bigcup_{n\in\omega}\{n\}\times U_n)^*$ we will mean the following subset of $X_{\gamma+1}$:

$$\mathrm{cl}_{\beta(\omega\times X_\gamma)}\left(\bigcup_{n\in\omega}\{n\}\times U_n\right)\cap X_{\gamma+1}.$$

Fact 1: $wn(P(X))=\omega_1$.

Proof. Take $\alpha\in\omega_1$ and let $\{C_n:n\in\omega\}$ be pairwise disjoint clopen subsets of X_α. By definition of $f_{\alpha+1,\alpha}$, if $C=[\bigcup_{n\in\omega}\{n\}\times C_n]^*$ then $f_{\alpha+1,\alpha}[C]$ is a nowhere dense subset of X_α because $f_{\alpha+1,\alpha}[C]\subset\overline{\bigcup C_n}\backslash\bigcup C_n$. It follows that $X_{\alpha+1}$, and also $P(X)$, has a π-base of clopen sets which map to nowhere dense subsets of X_α. For each $\alpha\in\omega_1$, let D_α be the complement in $P(X)$ of the union of a maximal cellular family of clopen sets which map to nowhere dense subsets of X_α. Clearly $\bigcup_{\alpha<\omega_1}D_\alpha$ is dense in $P(X)$.

Let us define T'. Take κ to be the ordinal $ht(T)+\omega_1$. We may suppose that T is isomorphic to a sub-tree T_1 of ${}^{<\kappa}c$ and that for each $t\in T_1$, $|\{\alpha\in c: t^\frown\langle\alpha\rangle\notin T_1\}|=c$. For convenience, let us suppose that $T=T_1$ and for $t\in T$, $[t]$ denotes the corresponding subset of X. For $t\in T$, $o(t)$ not a limit, define $\bar t$ to be the clopen set of $P(X)$, $\varphi_x^{\leftarrow}([t])$. For $t\in T_\sigma$, $o(t)$ a limit, define $\bar t=\mathrm{int}\bigcap\{\overline{t|\alpha}:\alpha<o(t),\ \alpha$ not a limit$\}$. It is clear that $\{\bar t:t\in T\}$ ordered by reverse inclusion is isomorphic to T and that for $t\in T_\sigma$, $\bar t\neq\emptyset$.

Now, if $t\in T$, then $\{\bar s: s$ is a successor of t in $T\}$ is a cellular family of clopen subsets of $\bar t$ which is maximal iff it is finite. To see this observe that if $\{[s_n]:n\in\omega\}$ are pairwise disjoint clopen subsets of $[t]$ then $C=(\bigcup_{n\in\omega}\{n\}\times[s_n])^*$ is a clopen subset of X_1 and $f_{1,0}(C)\cap[s]=\emptyset$ for each $s>t$, $s\in T$. If $t\in T_\sigma$ and $\{\bar s: s\in T\wedge s$ a successor of $t\}$ is not maximal in $\bar t$ then we extend it to a maximal cellular family of size c of clopen subsets of $\bar t$ labelled $\{\overline{t^\frown\langle\alpha\rangle}:\alpha\in c\}$ so as to agree on T. By Fact 1 and 2.3, for each $t\in T_\sigma$ and $\alpha\in c$ with $t^\frown\langle\alpha\rangle\notin T$, but with $\overline{t^\frown\langle\alpha\rangle}$ defined, we may choose a πn-tree for $\overline{t^\frown\langle\alpha\rangle}$ isomorphic to ${}^{<\omega_1}c$ and label these sets $\{\overline{t^\frown\langle\alpha\rangle^\frown g}: g\in{}^{<\omega_1}c\}$ preserving the tree isomorphism. Let $T'=T_\sigma\cup\{t^\frown\langle\alpha\rangle^\frown g: t\in T_\sigma,\ t^\frown\langle\alpha\rangle\notin T,\ \overline{t^\frown\langle\alpha\rangle}$ defined, and $g\in{}^{<\omega_1}c\}$. Clearly T' satisfies the prescribed tree condition. It remains only to check that $\{\bar{t}': t'\in T'\}$ is a π-base for $P(X)$.

Fact 2: If $b\in{}^{<\kappa}c$ is a maximal chain of T and $\operatorname{int}\bigcap\{\overline{b\upharpoonright\alpha}: \alpha\in o(b)\}\neq\emptyset$ then $b\in T_\sigma$.

Proof. Assume that $b\notin T_\sigma$ and therefore that $cf(o(b))\neq\omega$. Let $\gamma\in\omega_1$ be the minimum ordinal such that for some clopen set $C\subset X_\gamma$, $f^{\leftarrow}_{\omega_1,\gamma}(C)\subset\overline{b\upharpoonright\alpha}$ for each $\alpha\in o(b)$. Since T is a π-base for X, $\gamma>0$. By the construction of X_γ, there are $\gamma_n<\gamma$ and clopen subsets C_n of X_{γ_n} with $C=(\bigcup_{n\in\omega}\{n\}\times g^{\leftarrow}_{\gamma,\gamma_n}(C_n))^*$ (where we assume $\gamma_n+1=\gamma$ for each n and $g_{\gamma\gamma_n}$ is identity on $X_{\gamma n}$ if γ is a successor). Observe that for all but finitely many $n\in\omega$, $C_n\subset f^{\leftarrow}_{\gamma_n,0}([b\upharpoonright_\alpha])$ for each $\alpha\in o(b)$ for otherwise $C\not\subset f^{\leftarrow}_{\gamma,0}([b\upharpoonright\alpha])$ for some $\alpha\in o(b)$ since $cf(o(b))\neq\omega$. However this contradicts the minimality of γ since for some n, $f^{\leftarrow}_{\omega_1,\gamma_n}(C_n)\subset f^{\leftarrow}_{\omega_1,0}([b\upharpoonright_\alpha])=\overline{b\upharpoonright_\alpha}$ for each $\alpha\in o(b)$.

Now suppose $B\subset P(X)$ is a non-empty clopen set. If we find a $t'\in T'\backslash T$ with $B\cap\bar{t}'\neq\emptyset$ then for some $s\in T'$, $\bar{s}\subset B$ as T' contains a πn-tree for $\bar{t}'$. Take $b\in{}^{\kappa}c$, maximal with respect to $b\upharpoonright\alpha\in T$ for each $\alpha\in o(b)$ and $B\subset\bar{b}$. By Fact 2, $b\in T_\sigma$. Let $S_0=b$ and choose recursively for $n\in\omega$, if possible, $S_{n+1}\in T$ with $S_n<S_{n+1}$, $B\cap\bar{S}_{n+1}\neq\emptyset$ and $B\cap\bar{S}_n\backslash\bar{S}_{n+1}\neq\emptyset$.
Case 1: We have S_n but cannot choose S_{n+1}. By Fact 2, there is an $S\in T_\sigma$ with $\bar{S}\supset\bar{S}_n\cap B\notin\emptyset$ and for $t\in T$ with $S<t$, $\bar{t}\cap\bar{S}_n\cap B=\emptyset$. It follows that there is an $\alpha\in c$ with $s^\frown\langle\alpha\rangle\in T'\backslash T$ and $s^\frown\langle\alpha\rangle\cap B\neq\emptyset$.
Case 2: We have chosen S_n for $n\in\omega$. Let $\sup\{S_n: n\in\omega\}=s_\omega\in T_\sigma$ and $\alpha=o(s_\omega)$. For each $n\in\omega$, choose $t_n\in T$ so that $S_n<t_n$, $[t_n]\cap[S_{n+1}]=\emptyset$, and $\bar{t}_n\cap B\neq\emptyset$. Let $B_0=f^{\leftarrow}_{\omega_1,1}((\bigcup_{n\in\omega}\{n\}\times[t_n])^*)$.

Clearly $B_0\subset\bar{S}_n$ for each $n\in\omega$ and so $B_0\subset\bar{s}_\omega$. Also, since $f_{\omega_1,0}(B)\cap[t_n]\neq\emptyset$ for each n, $B_0\cap B\neq\emptyset$. Finally, if $s>s_\omega$, $s\in T$, then $\underline{(\omega\times[s])}\cap\bigcup(\{n\}\times[t_n])=\emptyset$ and therefore $\bar{s}\cap B_0=\emptyset$. Hence, for some $\alpha\in c$, $\overline{s_\omega{}^\frown\langle\alpha\rangle}\cap B_0\cap B\neq\emptyset$ with $s_\omega{}^\frown\langle\alpha\rangle\in T'\backslash T$ and we are done. $\square$

Proof of 4.1. Let T be an ω-complete branching tree of height and cardinality at most c. The cardinality of T is c because it is branching and ω-complete. We may assume that T is c-branching because it obviously is at limits of cofinality ω. Let $\kappa=ht(T)$ and assume that T is a full branching subtree of ${}^{<\kappa}(c\times c)$. Observe that if $t\in T$ and $s\in{}^{<\omega_1}(c\times c)$ then $t^\frown s\in T$. Fix an indexing $\{t_\alpha: \alpha\in c\}$ of T.

Let $X_0=(\omega\times 2^{\omega_1})^*$, $a_0=0$, and choose a πn-tree for X_0 isomorphic to ${}^{<\omega_1}(c\times\{0\})=T_0$ with elements on non-limit levels being clopen sets. It should cause no confusion if we think of elements of the trees we shall be defining as being both elements of ${}^{<\kappa}(c\times c)$ and as clopen subsets of the spaces we define. Suppose $\alpha\in c$ and for $\gamma<\alpha$ we have chosen an element $a_\gamma\in c$ and defined an inverse limit $\{X_\gamma: f_{\gamma\delta}\}$ of Parovičenko spaces with πn-trees T_γ such that:

(i) T_γ is a full branching ω-complete sub-tree of ${}^{<\kappa}(c\times\{a_\delta\colon \delta\le\gamma\})$.

(ii) T_γ is a subtree of T and if $\delta<\gamma$, the πn-tree T_δ pulls back into the πn-tree T_γ by $f_{\gamma,\delta}$ as a sub-tree and

(iii) if $\gamma=\delta+1$ and $\xi_\gamma<c$ is the smallest ordinal such that $t_{\xi_\gamma}|n\in T_\delta$ for each $n<o(t_{\xi_\gamma})$ and $t_{\xi_\gamma}\notin T_\delta$ then $t_{\xi_\gamma}\in T_\gamma$.

Let us construct X_α, T_α and $f_{\alpha,\gamma}$.

Case 1: cf $\alpha>\omega$. Let $X_\alpha=\underleftarrow{\lim}\{X_\gamma\colon \gamma<\alpha\}$, $T_\alpha=\bigcup_{\gamma<\alpha}T_\gamma$, $f_{\alpha\gamma}=\underleftarrow{\lim}\{f_{\beta\gamma}\colon \gamma<\beta<\alpha\}$ and $a_\alpha\in c\backslash\{a_\delta\colon\delta<\alpha\}$. X_α is a Parovičenko space since $cf(\alpha)>\omega$ and (i)–(iii) are obvious.

Case 2: $\alpha=\gamma+1$. Choose $\xi_\alpha\in c$ as in (iii) and take $a_\alpha\in c\backslash\{a_\delta\colon\delta<\alpha\}$ so that range $(t_{\xi_\alpha})\subset A_\alpha=\{a_\delta\colon\delta\le\alpha\}$. If $o(t_{\xi_\alpha})$ is a successor, let $X_\alpha=P(X_\gamma)$, $f_{\alpha\gamma}=\varphi_x$ and label T' obtained in 4.2 as a subtree of ${}^{<\kappa}(c\times A_\alpha)$ so as to be full-branching and extend T_γ. Obviously $t_{\xi\alpha}\in T_\alpha$ and $T_\alpha\subset T$ because $t\hat{\ }s\in T$ for each $t\in T$ and $s\in{}^{<\omega_1}(c\times c)$ and of the property of T'.

If $o(t_{\xi_\alpha})$ is not a successor then $cf(o(t_{\xi_\alpha}))>\omega$ since T_γ is ω-complete. Choose arbitrarily a point $x\notin X_\gamma$ and let $X=X_\gamma\cup\{x\}$ where x is isolated. We define a πn-tree $S_\gamma=T_\gamma\cup\{t_{\xi_\alpha}\}$ for X by simply adding x to all elements of T_γ which are less than t_{ξ_α} and letting t_{ξ_α} correspond to $\{x\}$. Let $X_\alpha=P(X)$ from 4.2 and $T_\gamma\cup\{t_{\xi_\alpha}\}$ embeds naturally into $T_\alpha=T'\subset{}^{<\kappa}(c\times A_\gamma)$. Properties (i)–(iii) follow easily from the construction of $P(X)$.

Case 3: $cf(\alpha)=\omega$. In this case let $X=\underleftarrow{\lim}\{X_\gamma\colon\gamma<\alpha\}$, $S=\bigcup_{\gamma<\alpha}T_\gamma$ and take $a_\alpha\in c\backslash\{a_\gamma\colon\gamma<\alpha\}$ arbitrarily. For $\gamma<\alpha$ let $g_{\alpha\gamma}=\underleftarrow{\lim}\{f_{\beta\gamma}\colon\gamma<\beta<\alpha\}$ and observe that the πn-tree T_γ pulls back naturally as a sub-tree of the πn-tree S for X. Let $X_\alpha=P(X)$ and label T' from 4.2 as a full-branching subtree of ${}^{<\kappa}(c\times\{a_\gamma\colon\gamma\le\alpha\})$ so as to extend S and call this tree T_α. Obviously, we let $f_{\alpha,\gamma}=g_{\alpha,\gamma}\circ\varphi_x$, for each $\gamma<\alpha$ and (i) to (iii) hold.

Finally let $X(T)=\underleftarrow{\lim}\{X_\alpha\colon\alpha<c\}$. It is clear that $T=\bigcup_{\alpha<c}T_\alpha$ and that T is a πn-tree for $X(T)$. $X(T)$ is a Parovičenko space because $cf(c)>\omega$ and each X_α is a Parovičenko space. □

It is now obvious that the investigation of co-absolutes of Parovičenko spaces should switch to an investigation of trees.

4.3. Definition. Let κ be a cardinal. A tree T *is κ-dense* if for any collection, $\mathscr{A}$, of maximal antichains of T with $|\mathscr{A}|\le\kappa$, $\bigcup\mathscr{A}$ is not dense in the tree $\{t\in T\colon t>s\}$ for any $s\in T$. ($D\subset T$ is *dense* if for each $t\in T$ there is a $d\in D$ with $d>t$). Call T *weakly κ-dense* if $\bigcup\mathscr{A}$ is not dense in T.

4.4. Lemma. *Let X be a Parovičenko space and let T be a πn-tree for X. Then $wn(x)=\min\{\kappa\colon T$ is not weakly κ-dense$\}$.*

Proof. It is clear that both cardinals, $wn(X)$ and $\min\{\kappa\colon T$ is not weakly κ–dense$\}$, are equal to $\min\{\kappa\colon$ there are κ open dense subsets of X whose intersection has empty interior$\}$.

We have the following characterization as an easy consequence of our previous results.

4.5. Theorem. *All Parovičenko spaces are co-absolute iff there is no branching ω-complete tree of cardinality c which is ω_1-dense.*

Proof. First assume that not all Parovičenko spaces are co-absolute. By 3.1, therefore, there is a Parovičenko space X with $wn(X) > \omega_1$ without loss of generality, $wn(U) > \omega_1$ for each regular open subset U of X. Now by 2.3 and 4.4, there is a branching ω-complete tree of cardinality c which is ω-dense. Conversely if T' is such a tree, then apply 4.1 and 4.4 to obtain a Parovičenko space X with $wn(X) > \omega_1$. Compare with the remark after Corollary 3.2.

4.6. Corollary. $cf(c) = \omega_1$ *implies there is no branching ω-complete tree or cardinality c which is ω_1-dense.*

4.7. Remark. One would like to determine if the converse to 4.6 is true. It has been shown that if it is consistent that there is an inaccessible cardinal then it is consistent that there is no branching ω-dense tree of cardinality ω_1 (independently by Davies [3] and Todorcevic [13]).

5. More on $wn(X)$

In this section we will show that the result in 3.1 will not generalize to ω_2 (at least, assuming a large cardinal exists). One might expect that if X and Y are Parovičenko spaces satisfying $wn(U) = \omega_2$ for each regular open subset U of $X \cup Y$, then X and Y should be co-absolute. However, as we shall see, this need not be the case.

To minimize the amount of work, we have chosen to use a model and a tree already in the literature. A combinatorial principle, $\square$, which holds in the constructible universe, is the following: there is a collection $\{C_\alpha : \alpha < \omega_2, \alpha \text{ a limit}\}$ such that for every limit $\alpha < \omega_2$,

(i) C_α is a closed unbounded subset of α;
(ii) if $cf\, \alpha = \omega$, then C_α is countable; and
(iii) if γ is a limit point of C_α, then $C_\gamma = C_\alpha \cap \gamma$.

It has been shown, assuming large cardinals, that there is a model in which $\square$ fails (see [8], p. 585). For the remainder of this section we shall assume that M is a model of $2^\omega = 2^{\omega_1} = \omega_2$ and $\square$ fails in M.

Define a tree T' as follows. The underlying set for T' is $\{\{C_\alpha : \alpha < \gamma\}$: where $\gamma < \omega_2$ is a limit ordinal, and the set $\{C_\alpha : \alpha < \gamma\}$ satisfies (i)–(iii) of $\square\}$. The ordering on T' is $\{C_\alpha : \alpha < \gamma\} \leq \{C'_\alpha : \alpha < \gamma'\}$ if $\gamma \leq \gamma'$ and, for $\alpha < \gamma$, $C_\alpha = C'_\alpha$. Now let $T \subset T'$ be the set of all $p \in T'$ such that p has a successor, with the same ordering. The tree, T, is defined in [8], p. 255.

It is shown in [8] that T is ω-closed and ω_1-dense. Indeed, it is very straightforward that T is ω-closed. To show that T is ω_1-dense, let $\{A_\alpha : \alpha \in \omega_1\}$ be maximal antichains of T. Let $P_0 \in T$ be arbitrary. Define recursively $P_\alpha \in T$, for $\alpha \in \omega_1$, as follows. Suppose we have chosen $\{P_\beta : \beta < \alpha\} \subset T$, an increasing chain satisfying: $P_{\beta+1}$ is greater than some element of A_β and for each limit ordinal $\xi < \alpha$, if $\gamma_\xi = \sup\{\gamma$: there is a $C_\gamma \in \bigcup\{P_\beta : \beta < \xi\}\}$, then there is a $C_{\gamma_\xi} \in P_\xi$ satisfying $C_{\gamma_\xi} \cap \gamma_\beta = C_{\gamma_\beta}$ for each limit $\beta < \xi$. It is obvious how to define P_α. If $\alpha = \beta + 1$, then choose $P_\alpha > P_\beta$ so that P_α is greater than some element of the maximal antichain A_β. If α is a limit, let $\gamma_\alpha = \sup\{\gamma$: there is a $C_\gamma \in \bigcup\{P_\beta : \beta < \alpha\}\}$. Now define C_{γ_α} so that it is closed unbounded in γ_α and so that $C_{\gamma_\alpha} \cap \gamma_\beta = C_{\gamma_\beta}$ for each limit $\beta < \alpha$.

Let $\gamma = \sup\{\gamma_\alpha : \alpha \in \omega_1\}$ and define $C_\gamma = \bigcup\{C_{\gamma_\alpha} : \alpha \in \omega_1\}$. It follows by the definition of the C_γ's that C_γ is a closed unbounded subset of γ. Also, if $\delta \in \gamma$ is a limit of C_γ, then for some $\alpha \in \omega_1$, δ is a limit of C_{γ_α}. Therefore $C_\gamma \cap \delta = C_{\delta_\alpha} \cap \delta = C_\delta$, where $C_\delta \in P_{\gamma_\alpha}$. Therefore $P = \bigcup_{\alpha \in \omega_1} P_\alpha \in T'$ and $P \cup \{C_\gamma\} \in T'$ is a successor of P, hence $P \in T$. It follows that T is ω_1-dense.

Clearly T is a branching ω-complete ω_1-dense tree of height and cardinality $c = 2^{\omega_1} = \omega_2$. Since $\square$ fails in M, T has no long branches ($= \omega_2$-branches). In contrast, let S be the complete binary tree of height ω_2. It is evident that T cannot be embedded densely in S.

Now, as in 4.1, let $X = X(S)$ and $Y = X(T)$. It is easily seen that X and Y do not have isomorphic πn-trees and so are not co-absolute. In fact, since S is ω_1-closed, non-empty intersections of descending chains of at most ω_1 clopen subsets of X have non-empty interior. Therefore X cannot be covered by ω_2 nowhere dense sets. In contrast, by 2.5 the Novák number of Y, $n(Y)$, is ω_2. However since both S and T are ω_1-dense, $wn(U) = \omega_2$ for any regular open $U \subset X \cup Y$.

Remark. An example of the above phenomenon can be obtained without large cardinals and in fact Y can be taken to be $\beta N \backslash N$ answering a question of [1]. Also there is an example of two non-co-absolute Parovičenko spaces with the same weak Novàk number and the same Novàk number. However both of these constructions require significantly more detailed forcing arguments and are done in [6].

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